

A note on Kirillov model for representations of $\mathrm{GL}_n(\mathbb{C})$

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Abstract

Let $G = \mathrm{GL}_n(\mathbb{C})$ and $1 \neq \psi : \mathbb{C} \rightarrow \mathbb{C}^\times$ be an additive character. Let U be the subgroup of upper triangular unipotent matrices in G . Denote by θ the character $\theta : U \rightarrow \mathbb{C}$ given by

$$\theta(u) := \psi(u_{1,2} + u_{2,3} + \dots + u_{n-1,n}).$$

Let P be the mirabolic subgroup of G consisting of all matrices in G with the last row equal to $(0, 0, \dots, 0, 1)$. We prove that if π is an irreducible generic representation of $\mathrm{GL}_n(\mathbb{C})$ and $\mathcal{W}(\pi, \psi)$ is its Whittaker model, then the space $\{f|_P : P \rightarrow \mathbb{C} : f \in \mathcal{W}(\pi, \psi)\}$ contains the space of infinitely differentiable functions $f : P \rightarrow \mathbb{C}$ which satisfy $f(up) = \psi(u)f(p)$ for all $u \in U$ and $p \in P$ and which have a compact support modulo U . A similar result was proven for $\mathrm{GL}_n(F)$, where F is a p -adic field by Gelfand-Kazhdan [1], and for $\mathrm{GL}_n(\mathbb{R})$ by Jacquet [2].

Let F be the field \mathbb{R} or \mathbb{C} . Let $G_n(F)$ be the group $\mathrm{GL}_n(F)$, let $P(F)$ be the mirabolic subgroup in $G_n(F)$ consisting of matrices in $G_n(F)$ with the last row equal to $(0, 0, \dots, 0, 1)$. Let $U_n(F)$ ($\overline{U}_n(F)$) respectively) be the unipotent subgroups consisting of upper triangular unipotent matrices (lower triangular unipotent matrices respectively) in $G_n(F)$. We fix a field F and will abbreviate G for the group $G(F)$. Let $\psi : F \rightarrow \mathbb{C}^\times$ be a non-trivial additive character of F and denote by θ_n a character on U_n given by

$$\theta_n(x) := \psi(x_{12} + x_{23} + \dots + x_{n-1,n})$$

for $x \in U$. Let π be a unitary irreducible representation of G_n on a Hilbert space \mathcal{H} with norm $\|\cdot\|$. We let \mathcal{V} be the space of the G_n smooth vectors, equipped with the topology defined by the semi-norms

$$v \rightarrow \|d\pi(X)v\|$$

with $X \in \mathcal{U}(G_n)$. We let \mathcal{V}' be the topological conjugate dual. We have inclusions

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'.$$

The positive-definite scalar product on $\mathcal{V} \times \mathcal{V}$ extends to $\mathcal{V} \times \mathcal{V}'$. Recall π is said to be generic if there is an element $\lambda \neq 0$ of \mathcal{V}' such that

$$(\pi(u)v, \lambda) = \theta_n(u)(v, \lambda)$$

for all $v \in \mathcal{V}$ and $u \in U_n$. Up to a scalar factor, the vector λ is unique [4]. For every $v \in \mathcal{V}$ we define a function W_v on G_n by

$$W_v(g) = (\pi(g)v, \lambda).$$

We let $\mathcal{W}(\pi, \psi)$ be the Whittaker model of π , that is, the space spanned by the functions

$$g \mapsto (\pi(g)v, \lambda),$$

where v is in \mathcal{V} . We identify \mathcal{V} and $\mathcal{W}(\pi, \psi)$. Then the scalar product induced by \mathcal{H} on \mathcal{V} is a scalar multiple of the scalar product defined by the convergent integral

$$(v_1, v_2) := \int_{U_{n-1} \backslash G_{n-1}} W_{v_1} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \overline{W_{v_2} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}} dg,$$

see [3]. We may assume the scalar product on \mathcal{V} is equal to this convergent integral.

We denote by $C_c^\infty(\theta_{n-1}, G_{n-1})$ the space of smooth functions $f : G_{n-1} \rightarrow \mathbb{C}$ which are compactly supported modulo U_{n-1} , such that $f(ug) = \theta_n(u)f(g)$ for all $u \in U_{n-1}$, $g \in G_{n-1}$.

In this note we prove the following theorem for $F = \mathbb{C}$.

Theorem 1. *Let π be a generic unitary irreducible representation of $G_n(:= G_n(\mathbb{C}))$. Given $\phi \in C_c^\infty(\theta_{n-1}, G_{n-1})$ there is a unique $W_\phi \in \mathcal{V}$ such that, for all $g \in G_{n-1}$,*

$$W_\phi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \phi(g).$$

Furthermore, the map $\phi \rightarrow W_\phi$ is continuous.

An analogous result was proven for $G_n(F)$, F a p -adic field by Gelfand-Kazhdan [1]. Recently it was proven by Jacquet for $G_n(\mathbb{R})$ [2]. Our proof for $G_n(\mathbb{C})$ is similar to Jacquet proof. We sketch here the main steps of the proof. There is only one technical lemma that should be reproved for $G_n(\mathbb{C})$.

For a Lie algebra \mathfrak{g} over \mathbb{C} we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. For a Lie group G we denote by $\text{Lie}(G)$ its Lie algebra (over \mathbb{C}), by $\mathcal{U}(G)$ its universal enveloping algebra, by $Z(\mathcal{U}(G))$ the center of $\mathcal{U}(G)$. Denote the image of $X \in \text{Lie}(G)$ under the natural embedding in $\mathcal{U}(G)$ by D_X . We will identify $\text{Lie}(G)$ with its image in $\mathcal{U}(G)$. Note that D_X can be realized as the linear differential operator on the space $C_c^\infty(G)$ corresponding to derivative in the direction X .

Let us denote the elementary matrix with 1 in the (i, j) place and 0 in all other places by E_{ij} . Let us denote $D_{E_{ij}} := D_{(i,j)}$. Denote the elementary matrix with $\sqrt{-1}$ in the place (i, j) and 0 in all other places by $D_{\sqrt{-1}(i,j)}$. The set

$$\left\{ D_{ij}, D_{\sqrt{-1}(i,j)} : 1 \leq i, j \leq n \right\}$$

is a basis of $\text{Lie}(G_n(\mathbb{C}))$. Observe that there are two natural embeddings of Lie algebras

$$i_1, i_2 : \text{Lie}(G_n(\mathbb{R})) \rightarrow \text{Lie}(G_n(\mathbb{C}))$$

given by

$$i_1(D_X) = \frac{1}{2} (D_X + \sqrt{-1} D_{\sqrt{-1}X}), \quad i_2(D_X) = \frac{1}{2} (D_X - \sqrt{-1} D_{\sqrt{-1}X}).$$

We have

$$\text{Lie}(G_n(\mathbb{C})) = i_1(\text{Lie}(G_n(\mathbb{R}))) \oplus i_2(\text{Lie}(G_n(\mathbb{R}))).$$

That is, the complex vector space $\text{Lie}(G_n(\mathbb{C}))$ is a direct sum of the vector spaces

$$i_1(\text{Lie}(G_n(\mathbb{R}))), i_2(\text{Lie}(G_n(\mathbb{R}))),$$

and for all $X, Y \in \text{Lie}(G_n(\mathbb{R}))$ we have $[i_1(X), i_2(Y)] = 0$.

The following lemma is the key lemma in the proof of Theorem 1 for $G_n(\mathbb{R})$.

Lemma 1. *Consider a $\text{Lie}(G_n(F))$ module V and a vector $0 \neq \lambda \in V$. Suppose $D_{ij}\lambda = 0$ for every pair of indices $1 \leq i, j \leq n$ such that $j > i + 1$. Suppose also that for every $1 \leq i \leq n - 1$ there is a constant $c_i \neq 0$ such that $D_{i, (i+1)}\lambda = c_i\lambda$. If $F = \mathbb{C}$ suppose also that $D_{\sqrt{-1}i, j}\lambda = 0$ for every pair of indices $1 \leq i, j \leq n$ such that $j > i + 1$ and for every $1 \leq i \leq n - 1$ there is a constant $c_{\sqrt{-1}i} \neq 0$ such that $D_{\sqrt{-1}(i, i+1)}\lambda = c_{\sqrt{-1}i}\lambda$. Then*

$$\begin{aligned} \text{Lie}(\overline{U_n(F)})\lambda &\subset \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F)))\lambda, \\ \mathcal{U}(\overline{U_n(F)})\lambda &\subset \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F)))\lambda, \\ \mathcal{U}(\overline{G_n(F)})\lambda &\subset \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F)))\lambda. \end{aligned}$$

Proof. We will use the corresponding result for the case $\text{Lie}(G_n(\mathbb{R}))$ and deduce from it the result for $\text{Lie}(G_n(\mathbb{C}))$. Note that it is enough to prove the first inclusion, as the second inclusion and the third inclusion follows from the first by an application of theorem of Poincare-Birkhoff-Witt. See [2], the first lines of the proof of Lemma 3.

To prove the first inclusion we have to prove that

$$D_X\lambda \in \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F)))\lambda$$

for

$$X \in \{E_{n1}, E_{n2}, \dots, E_{nn}\} \cup \left\{E_{\sqrt{-1}(n,1)}, E_{\sqrt{-1}(n,2)}, \dots, E_{\sqrt{-1}(n,n)}\right\}.$$

By the corresponding result for $\text{Lie}(G_n(\mathbb{R}))$ we know that for the Lie algebras $i_1(\text{Lie}(G_{n-1}(\mathbb{R})))$ and $i_2(\text{Lie}(G_{n-1}(\mathbb{R})))$ and $X \in \{E_{n1}, E_{n2}, \dots, E_{nn}\}$ we have

$$D_{i_{1,2}(X)}\lambda \in \mathcal{U}(i_{1,2}(\text{Lie}(G_{n-1}(\mathbb{R})))) Z(\mathcal{U}(i_{1,2}(\text{Lie}(G_{n-1}(\mathbb{R})))) \lambda \subset \mathcal{U}(G_{n-1}(\mathbb{C})) Z(\mathcal{U}(G_{n-1}(\mathbb{C}))) \lambda.$$

Thus,

$$\frac{1}{2}(D_{n,r}\lambda + \sqrt{-1}D_{\sqrt{-1}n,r}\lambda), \frac{1}{2}(D_{n,r} - \sqrt{-1}D_{\sqrt{-1}n,r}\lambda) \in \mathcal{U}(G_{n-1}(\mathbb{C})) Z(\mathcal{U}(G_{n-1}(\mathbb{C}))) \lambda.$$

By adding and subtracting the two terms on the left hand-side of the last formula we obtain

$$D_{n,r}\lambda, D_{\sqrt{-1}n,r}\lambda \in \mathcal{U}(G_{n-1}(\mathbb{C})) Z(\mathcal{U}(G_{n-1}(\mathbb{C}))) \lambda.$$

The lemma is proved. \square

For the convenience of a reader we rewrite here only formulations of the lemmas which are needed to prove Theorem 1. All the proofs are identical to that written in [2], so we see no need to repeat the proofs here.

From now on $F = \mathbb{C}$. Consider the space \mathcal{V}_{n-1} of G_{n-1} smooth vectors in \mathcal{H} .

Lemma 2. *We have continuous inclusions*

$$\mathcal{V} \subset \mathcal{V}_{n-1} \subset \mathcal{H} \subset \mathcal{V}'_{n-1} \subset \mathcal{V}'.$$

Lemma 3. *The vector λ which is a priori in \mathcal{V}' belongs to \mathcal{V}'_{n-1} .*

Given $\phi \in C_c^\infty(G_{n-1})$, we set

$$u_\phi := \int_{g \in G_{n-1}} \phi(g) \pi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \lambda dg.$$

Lemma 4. *The vector u_ϕ belongs to \mathcal{V}_{n-1} , in particular to \mathcal{H} . The map*

$$\phi \mapsto u_\phi, C_c^\infty(G_{n-1}) \rightarrow \mathcal{H}$$

is continuous.

Lemma 5. *For every $X \in \mathcal{U}(G_n)$ and every $\phi \in C_c^\infty(G_{n-1})$ the vector u_ϕ , which, a priori, is in \mathcal{V}' , is in fact in \mathcal{H} . Moreover, there is a continuous semi-norm μ on $C_c^\infty(G_{n-1})$ such that, for every $\phi \in C_c^\infty(G_{n-1})$,*

$$\|d\pi(X)u_\phi\| \leq \mu(\phi).$$

We note that Lemma 1 is used to prove the last Lemma.

Lemma 6. *Suppose $v_0 \in \mathcal{H}$ is a vector such that for every $X \in \mathcal{U}(G_n)$ the vector $d\pi(X)v_0$, which a priori is in \mathcal{V}' , is in fact in \mathcal{H} . Then v_0 is in \mathcal{V} .*

Proposition 1. *For every $\phi \in C_c^\infty(G_{n-1})$ the vector*

$$u_\phi := \int_{g \in G_{n-1}} \phi(g) \pi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \lambda dg$$

is in \mathcal{V} . Furthermore, the map

$$\phi \mapsto u_\phi, C_c^\infty(G_{n-1}) \rightarrow \mathcal{V}$$

is continuous.

Lemma 7. *If $v = u_\phi$ then*

$$W_v \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \phi_0(g),$$

where

$$\phi_0(g) = \int_{U_{n-1}} \phi((gu)^{-1}) \overline{\theta_{n-1}}(u) du.$$

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